ON THE MONODROMY ACTION ON MILNOR FIBERS OF GRAPHIC ARRANGEMENTS

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ABSTRACT. We analyze the monodromy action, over the rationals, on the first homology group of the Milnor fiber, for arbitrary subarrangements of Coxeter arrangements. We propose a combinatorial formula for the monodromy action, involving Aomoto complexes in positive characteristic. We verify the formula, in cases A, B and D.

1. Introduction and statement of results

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of complex hyperplanes in $\mathbb{C}^l$, with *complement* $M_{\mathcal{A}} = \mathbb{C}^l \setminus \bigcup_{i=1}^n H_i$, and *intersection lattice* $\mathcal{L}(\mathcal{A})$, consisting of the various intersections of hyperplanes from $\mathcal{A}$, ordered by reverse inclusion. A fundamental result in arrangement theory, due to Orlik and Solomon [17], relates the topology and the combinatorics of $\mathcal{A}$, by saying that the homology of $M_{\mathcal{A}}$ with arbitrary untwisted coefficients is *combinatorial*, i.e., is determined by the intersection lattice. More precisely, they proved that the cohomology ring with arbitrary coefficients, $H^*(M_{\mathcal{A}}, \Bbbk)$, is isomorphic to the Orlik-Solomon algebra of $\mathcal{A}$ over $\Bbbk$, $A^*_{\Bbbk}(\mathcal{A})$, which depends only on the lattice $\mathcal{L}(\mathcal{A})$.

Assuming $\mathcal{A}$ to be central, with homogeneous defining polynomial, $f_{\mathcal{A}}$, there is a well-known global Milnor fibration, $F_{\mathcal{A}} \hookrightarrow M_{\mathcal{A}} \xrightarrow{f_{\mathcal{A}}} \mathbb{C}^*$, where $F_{\mathcal{A}} := f_{\mathcal{A}}^{-1}(1)$ is the *Milnor fiber*. Milnor fibers of polynomials and their homology, especially the structure of the *monodromy action* on homology, play a key role in singularity theory, see for instance [8] and the references therein. An important problem in arrangement theory is to decide whether $H_*(F_{\mathcal{A}}, \mathbb{Q})$ is combinatorially determined. To our best knowledge, the problem is open, even in degree $* = 1$. (Libgober's

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description [14, 15] of the monodromy action, in terms of superabundance of curves, is apriori non-combinatorial.)

The finite *graphs* Γ we consider in this paper, with vertex set V and edges E, have at most double edges connecting two distinct vertices, and at most one loop at each point. The presence of a loop at i will be denoted by $\odot\, i$. Edges are labeled with signs: double edges are indicated by the label $\pm$, positive simple edges by $+$, and the absence of a label indicates a negative edge.

An *unsigned graph* means an ordinary finite simplicial graph (with no double edges or loops), where all edges are negative. A *signed graph* is a graph without loops. The graphs Γ we are considering here encode subarrangements of Coxeter arrangements of type B, called *graphic arrangements* and denoted by $\mathcal{A}(\Gamma)$. The signed graphs correspond to subarrangements of Coxeter arrangements of type D, while the unsigned ones parametrize type A subarrangements. The definition of $\mathcal{A}(\Gamma)$ is the obvious one; see Definition 4.3.

For example, in the figure below Γ is unsigned, whereas Γ' has a double edge, 5 negative edges, 4 positive edges, and 3 loops.

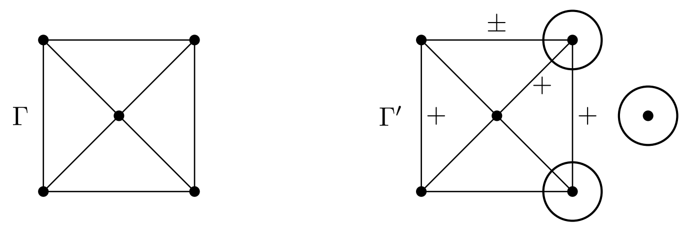


FIGURE 1. Two graphs

Since the geometric monodromy action on $F_{\mathcal{A}}$ has order n, it follows that one has an equivariant decomposition (with respect to the homology monodromy action),

$$H_q(F_{\mathcal{A}}, \mathbb{Q}) = \bigoplus_{d|n} \Big(\frac{\mathbb{Q}[t]}{\Phi_d}\Big)^{b_{qd}(\mathcal{A})} \tag{1.1}$$

for all q, where Φ_d is the dth cyclotomic polynomial; see [18, 13].

The numbers $b_{q1}(\mathcal{A})$, $q \geq 0$, are combinatorially determined, being equal to the corresponding Betti numbers of the associated projective arrangement $\overline{\mathcal{A}}$; see [18]. In particular, $b_{11}(\mathcal{A}) = n - 1$. We may also assume in (1.1) that $r := \mathrm{rk}(\mathcal{A}) \geq 3$ (if $r = 1$, $F_{\mathcal{A}}$ is a point, and the rank 2 case is treated in [18, Proposition 5.125]).

Our main result in this paper establishes a combinatorial formula for the numbers $b_d(\Gamma) := b_{1d}(\mathcal{A}(\Gamma))$, in the case of graphic arrangements. To describe it,

we need to recall the general definition of *Aomoto complexes* associated to Orlik-Solomon algebras, $A^*_{\Bbbk}(\mathcal{A})$. Let $\omega \in A^1_{\Bbbk}(\mathcal{A})$ be an arbitrary element. Since A^* is a quotient of an exterior algebra, the square $\omega \cdot \omega$ vanishes. Denoting by μ_ω left-multiplication by ω in A^*, we thus obtain a cochain complex,

$$\big(A^*_{\Bbbk}(\mathcal{A}), \mu_\omega\big) = \{A^*_{\Bbbk}(\mathcal{A}) \xrightarrow{\mu_\omega} A^{*+1}_{\Bbbk}(\mathcal{A})\}_{*\geq 0}\,, \tag{1.2}$$

called the Aomoto complex of ω, introduced by Aomoto in [1], and studied by Falk in [12], from the point of view of resonance varieties of arrangements.

By definition, $A^1_{\Bbbk}(\mathcal{A})$ is freely generated by $\{a_H\}_{H\in\mathcal{A}}$. So, $\omega = \sum_{H\in\mathcal{A}} \lambda_H a_H$, with $\lambda_H \in \Bbbk$. Denote by ω_1 the distinguished element $\omega_1 := \sum_H a_H$, and abbreviate μ_{ω_1} by μ_1. Let $\Bbbk$ be a field, $\operatorname{char} \Bbbk = p$. Set

$$\beta_{qp}(\mathcal{A}) := \dim_{\Bbbk} H^q(A^*_{\Bbbk}(\mathcal{A}), \mu_1) \quad \text{for} \quad q \geq 0\,. \tag{1.3}$$

One knows [26] that $\beta_{q0}(\mathcal{A}) = 0$, for all q. When $\mathcal{A} = \mathcal{A}(\Gamma)$ is a graphic arrangement, set $\beta_p(\Gamma) := \beta_{1p}(\mathcal{A}(\Gamma))$, for each prime p.

The off-diagonal elements different from 2 of type $A - I$ Coxeter matrices, in rank ≥ 3, are 3, 4 and 5 [2]. All of them are of the form p^k, with $p \in \{2, 3, 5\}$. The theorem below relates the numbers $b_d(\Gamma)$ from (1.1) to the numbers $\beta_p(\Gamma)$ coming from (1.2).

Theorem A. *Let $\mathcal{A}(\Gamma)$ be an arbitrary graphic arrangement of rank at least 3, with n hyperplanes, and let $d \neq 1$ be a divisor of n.*

1. *If $d \neq 3$, then $b_d(\Gamma) = 0$.*
2. *If $p \neq 3$ is prime, then $\beta_p(\Gamma) = 0$.*
3. *If $n \equiv 0$ (mod 3), then $b_3(\Gamma) = \beta_3(\Gamma)$. If $n \not\equiv 0$ (mod 3), then $\beta_3(\Gamma) = 0$.*
4. *The following formula holds for the Milnor fiber F_Γ:*

$$H_1(F_\Gamma, \mathbb{Q}) = \big(\frac{\mathbb{Q}[t]}{t-1}\big)^{n-1} \oplus \big(\frac{\mathbb{Q}[t]}{\Phi_2} \oplus \frac{\mathbb{Q}[t]}{\Phi_4}\big)^{\beta_2(\Gamma)} \oplus \big(\frac{\mathbb{Q}[t]}{\Phi_3}\big)^{\beta_3(\Gamma)} \oplus \big(\frac{\mathbb{Q}[t]}{\Phi_5}\big)^{\beta_5(\Gamma)}\,.$$

We conjecture that the above formula (4) actually holds for all subarrangements $\mathcal{A}$ of rank at least 3 of arbitrary Coxeter arrangements.

A useful fact is that the $\mathbb{Q}[t]$–module structure of $H_*(F_{\mathcal{A}}, \mathbb{Q})$ depends only on the lattice-isotopy type (in the sense of Randell [21]) of the arrangement $\mathcal{A}$; see Section 2 for more details. With this remark, (1.1) takes the following explicit form, when $\mathcal{A}$ is graphic.

Theorem B. *Let $\mathcal{A}(\Gamma)$ be an arbitrary graphic arrangement of rank at least 3, with Milnor fiber F_Γ. Set $n := |\mathsf{E}(\Gamma)|$.*

(1) *If $\mathcal{A}(\Gamma)$ is lattice-isotopic to either D_3 or D_4 (the full Coxeter arrangements of type D and rank 3 or 4), then*

$$H_1(F_\Gamma, \mathbb{Q}) = \big(\frac{\mathbb{Q}[t]}{t-1}\big)^{n-1} \oplus \big(\frac{\mathbb{Q}[t]}{t^2+t+1}\big)\,.$$

(2) *Otherwise, $H_1(F_\Gamma, \mathbb{Q}) = \big(\frac{\mathbb{Q}[t]}{t-1}\big)^{n-1}$.*

Similar results (proving the asymptotic triviality of the monodromy action on $H_q(F_\Gamma, \mathbb{Q})$) have been obtained by Settepanella, in the particular case of complete graphic arrangements on $v \gg q$ vertices, of types A, B and D; see [23].

However, the methods are completely different. The main tool from [23] is the Salvetti complex associated to a Coxeter group. This technique does not seem to extend to arbitrary subarrangements of Coxeter arrangements. Our strategy is to use the known relationship between Milnor fibers and twisted homology, see for instance Cohen-Suciu [5]. To compute the latter, via Aomoto complexes, we rely on three key results: the first in characteristic zero ([11, 22]), the second in arbitrary characteristic ([26]), and the last in positive characteristic ([3, 20]). These techniques are available for arbitrary arrangements $\mathcal{A}$.

Based on a method due to Deligne [7], Esnault-Schechtman-Viehweg [11] and Schechtman-Terao-Varchenko [22] showed that twisted homology on $M_{\mathcal{A}}$ may be computed by Aomoto complexes in characteristic zero, for certain local systems. Unfortunately, this approach does not always work, see e.g. Example 3.12. When the Deligne method is available, it may be combined with general results on Aomoto complexes, due to Yuzvinsky [26], to obtain vanishing results. We use this approach in Theorem A (1), for $d \neq 2, 3, 4$.

To settle the remaining cases, we resort to *modular upper bounds*, for the dimension over $\mathbb{C}$ of twisted homology with rational local systems whose denominator is a prime power, p^k. Improving a result due to Cohen and Orlik [3] for $k = 1$, it is shown in [20] that these dimensions are bounded above by numbers coming from objects in characteristic p; in the equimonodromical case, these numbers are defined by (1.3). This method yields Theorem B (2).

In all previously known (sporadic) examples, the modular inequalities become equalities, for equimonodromical rational local systems with $k = 1$; see [4, Section 7]. We may add the following new large class of examples to the list.

Theorem C. *Let $\mathcal{A}(\Gamma)$ be a graphic arrangement (of arbitrary rank). The modular upper bound for equimonodromical rational local systems on $M_{\mathcal{A}(\Gamma)}$ with denominator p is equal to the dimension of the corresponding twisted homology in degree one, for every prime p.*

Our approach also leads to a partial verification of formula (4) from Theorem A, for arbitrary subarrangements of arbitrary Coxeter type; see Corollary 3.15.

2. Homology of Milnor fibers and twisted homology

In this section, we will review the relationship between the cyclotomic decomposition of $H_*(F_{\mathcal{A}}, \mathbb{Q})$, and the (co)homology of the complement of $\mathcal{A}$ with coefficients in rank one local systems.

Assume $\mathcal{A}$ is an arrangement in $\mathbb{C}^l$, defined as the zero set of the homogeneous polynomial $f_{\mathcal{A}}$. There is an action on $\mathbb{C}^l$, given by the multiplication with $u = \exp \frac{2\pi\sqrt{-1}}{n}$, where $n = |\mathcal{A}|$, which induces an action on the fiber $F_{\mathcal{A}}$ (since $f_{\mathcal{A}}$ is homogeneous of degree n). We call this action on the Milnor fiber the *geometric monodromy*, denoted by $h : F_{\mathcal{A}} \longrightarrow F_{\mathcal{A}}$. The induced action on homology, $h_*\colon H_*(F_{\mathcal{A}}, \mathbb{Q}) \to H_*(F_{\mathcal{A}}, \mathbb{Q})$, corresponds to multiplication by t, in equation (1.1).

2.1. This may be conveniently reinterpreted in terms of *twisted homology*, as follows. The complement $M_{\mathcal{A}}$ is a connected, 1–*marked*, finite type CW–space. That is, it is endowed with a $\mathbb{Z}$–basis of $H_1(M_{\mathcal{A}})$, denoted by $\{a_H^*\}_{H\in\mathcal{A}}$, dual to the canonical basis of $A^1_{\mathbb{Z}}(\mathcal{A})$. The marking defines a $\mathbb{Z}$–character, $\nu\colon H_1(M_{\mathcal{A}}) \to \mathbb{Z}$, which sends each a_H^* to 1. This character induces on group rings a homomorphism, $\nu\colon \mathbb{Z}\pi_1(M_{\mathcal{A}}) \to \mathbb{Z}[t^{\pm 1}]$, which gives rise to a $\mathbb{Z}\pi_1(M_{\mathcal{A}})$–module (alias a local system on $M_{\mathcal{A}}$), denoted by $\mathbb{Q}[t^{\pm 1}]_\nu$. There is an equivariant isomorphism

$$H_*(F_{\mathcal{A}}, \mathbb{Q}) \cong H_*(M_{\mathcal{A}}, \mathbb{Q}[t^{\pm 1}]_\nu)\,, \tag{2.1}$$

see [8, p.106–107] and [25, Ch.VI].

2.2. One may consider arbitrary ring homomorphisms $\nu\colon \mathbb{Z}\pi_1(M_{\mathcal{A}}) \to R$, where R is a commutative ring, with group of units R^*. These morphisms are naturally identified with elements of $\operatorname{Hom}(H_1(M_{\mathcal{A}}), R^*) \equiv (R^*)^n$. The associated local system, R_ν, is called *equimonodromical* if ν is constant on the distinguished basis $\{a_H^*\}$. It follows from [19, p.497-498] that the equivariant isomorphism type of $H_*(M_{\mathcal{A}}, R_\nu)$ depends only on the lattice-isotopy type of $\mathcal{A}$, in the equimonodromical case. From the definitions, we also see that the cochain isomorphism type of the Aomoto complex $(A^*_{\Bbbk}(\mathcal{A}), \mu_1)$ defined in the Introduction depends only on lattice-isotopy type.

2.3. Twisted homology with coefficients in rank one local systems, $H_*(M_{\mathcal{A}}, \mathbb{C}_\rho)$, is a very active research area in arrangement theory. Here, $\rho \in \operatorname{Hom}(H_1(M_{\mathcal{A}}), \mathbb{C}^*)$ denotes an arbitrary *character*. The *rational* characters play an important role.

Definition 2.4. Let $\mathbf{k} = (k_H)_{H\in\mathcal{A}}$ be a collection of integers, with g.c.d. equal to 1. Let $u \in \mathbb{C}^*$ be a primitive d–root of unity. The character ρ defined by $\rho(a_H^*) = u^{k_H}$ is called rational. If $\mathbf{k} = \mathbf{1}$, ρ is called rational and equimonodromical, with denominator d.

Set $b_q(\mathcal{A}, \frac{\mathbf{k}}{d}) := \dim_{\mathbb{C}} H_q(M_{\mathcal{A}}, \mathbb{C}_\rho)$. (This is well-defined, by Galois theory.) As is well-known (see e.g. [9]), one has the following recurrence formula, for $d \mid n$:

$$b_q(\mathcal{A}, \frac{\mathbf{1}}{d}) = b_{qd}(\mathcal{A}) + b_{q-1,d}(\mathcal{A})\,, \forall q\,. \tag{2.2}$$

In particular, $b_d(\mathcal{A}) := b_{1d}(\mathcal{A}) = b_1(\mathcal{A}, \frac{\mathbf{1}}{d})$, for $1 \neq d \mid n$.

2.5. We close this section by describing a method for computing twisted homology on $M_{\mathcal{A}}$, by using *generic sections.*

We will need the following version of twisted Betti numbers, for arbitrary Aomoto complexes. Given $\omega \in A^1_{\Bbbk}(\mathcal{A})$, $\Bbbk$ a field, set

$$\beta_q(\mathcal{A}, \omega) := \dim_{\Bbbk} H^q(A^{\bullet}_{\Bbbk}(\mathcal{A}), \mu_\omega) \quad \text{for} \quad q \geq 0\,. \tag{2.3}$$

We may now spell out our result.

Proposition 2.6. *Let $\mathcal{A}$ be a rank $r \geq 3$ arrangement in $\mathbb{C}^l$. Let $U \subset \mathbb{C}^l$ be a subspace of dimension $k+1$, $2 \leq k < r$. Denote by $\mathcal{A}^U$ the restriction, and by $j\colon M_{\mathcal{A}} \cap U \hookrightarrow M_{\mathcal{A}}$ the inclusion map between complements. If U is $\mathcal{L}_k(\mathcal{A})$–generic, in the sense of* [10, §5(1)]*, the following hold.*

1. *The map induced by j on π_1 is an isomorphism, preserving the natural 1–markings upon abelianization.*
2. *The map induced on Aomoto complexes, $j^*\colon (A^*_{\Bbbk}(\mathcal{A}), \mu_\omega) \to (A^*_{\Bbbk}(\mathcal{A}^U), \mu_\omega)$, is an isomorphism for $* \leq k$. In particular, $\beta_q(\mathcal{A}, \omega) = \beta_q(\mathcal{A}^U, \omega)$, for any ω and every $q < k$.*
3. *The map induced on twisted homology, $j_*\colon H_*(M_{\mathcal{A}} \cap U, j^*R) \to H_*(M_{\mathcal{A}}, R)$, is an isomorphism for $* < k$ and an epimorphism for $* = k$, for arbitrary coefficients. Moreover, $j^*R \equiv R$, if R comes from a representation, $\nu\colon \mathbb{Z}\pi_1(M_{\mathcal{A}}) \to R$, where R is a commutative ring.*

Proof. By [10, Proposition 5.14], j induces an isomorphism on π_q, for $q < k$, and a surjection on π_k.

(1) Remember that $k \geq 2$, to obtain the assertion on π_1. The claim on markings is obvious. Put together, these two properties show that $j^*R \equiv R$, if R comes from an abelian representation.

(2) Follows from the fact that $\mathcal{A}^U$ and $\mathcal{A}$ have the same dependent subarrangements, up to cardinality $k+1$.

(3) The first claim is a standard consequence of the properties of $j_\sharp$ on $\pi_{\leq k}$, see [25, Ch.VI], and the second was already clarified in the proof of (1). □

We will prove that, for almost all graphic arrangements, the only nontrivial component from decomposition (1.1) in degree 1 is the part corresponding to Φ_1. To do this, we turn to combinatorial computations.

3. Twisted homology and Aomoto complexes

Let $\omega \in A^1_{\mathbb{C}}(\mathcal{A})$ be a degree one element of the Orlik–Solomon algebra of $\mathcal{A}$ with complex coefficients. Write $\omega = \sum_{H\in\mathcal{A}} \lambda_H a_H$, with $\lambda_H \in \mathbb{C}$. Consider the character torus, $\mathbb{T}_{\mathcal{A}} := \mathrm{Hom}(\pi_1(M_{\mathcal{A}}), \mathbb{C}^*) = \mathrm{Hom}(H_1(M_{\mathcal{A}}), \mathbb{C}^*) \equiv (\mathbb{C}^*)^n$, and the rank one complex local system associated to ω, $\rho_\omega := (\exp(2\pi\sqrt{-1}\lambda_H))_{H\in\mathcal{A}} \in \mathbb{T}_{\mathcal{A}}$. Clearly, $\rho_\omega = \rho_{\omega+\alpha}$, for all $\alpha \in \mathbb{Z}^n$.

3.1. Basic results from [11, 22] establish a deep connection between the twisted cohomology of $M_{\mathcal{A}}$, $H^*(M_{\mathcal{A}}, {}_{\rho_\omega}\mathbb{C})$, and the cohomology of the Aomoto complex of ω, $(A^\bullet_{\mathbb{C}}(\mathcal{A}), \mu_\omega)$, for *nonresonant* ω.

Definition 3.2. An element $X \in \mathcal{L}(\mathcal{A})$ is called *dense* if the arrangement $\mathcal{A}_X$ is not decomposable as a nontrivial product.

Example 3.3. *(i)* All hyperplanes are dense elements.
(ii) An element X of rank 2 is dense if and only if $|\mathcal{A}_X| \geq 3$.

For $X \in \mathcal{L}(\mathcal{A})$, set $m_X := |\mathcal{A}_X|$. For $\omega = \sum_{H\in\mathcal{A}} \lambda_H a_H \in A^1_{\Bbbk}(\mathcal{A})$ and $X \in \mathcal{L}(\mathcal{A})$, set $\omega_X := \sum_{H\supset X} \lambda_H a_H \in A^1_{\Bbbk}(\mathcal{A}_X)$, and $\Sigma_X\omega := \sum_{H\supset X} \lambda_H \in \Bbbk$. For a central arrangement $\mathcal{A}$, let $C := \cap_{H\in\mathcal{A}} H$ be the center of $\mathcal{A}$.

Definition 3.4. Let $\mathcal{A}$ be a central arrangement. An element $\omega = \sum_{H\in\mathcal{A}} \lambda_H a_H \in A^1_{\mathbb{C}}(\mathcal{A})$ is called *nonresonant* if $\Sigma_X\omega \notin \mathbb{Z}_{>0}$, for all dense elements $X \in \mathcal{L}(\mathcal{A})$, and $\Sigma_C\omega \notin \mathbb{Z}_{<0}$.

One may reduce the computation of twisted homology to a combinatorial problem, under a nonresonance assumption, via the following result.

Theorem 3.5 ([11, 22])**.** *Let $\omega \in A^1_{\mathbb{C}}(\mathcal{A})$ be a nonresonant element. Then*

$$\dim_{\mathbb{C}} H_q(M_{\mathcal{A}}, \mathbb{C}_{\rho_\omega}) = \beta_q(\mathcal{A}, \omega)\,, \forall q\,.$$

3.6. We define now a partial nonresonance condition.

Definition 3.7. An element $\omega = \sum_{H\in\mathcal{A}} \lambda_H a_H \in A^1_{\mathbb{C}}(\mathcal{A})$ is called *k–nonresonant* ($k \geq 1$), if $\Sigma_X \omega \notin \mathbb{Z}_{>0}$, for all dense elements $X \in \mathcal{L}(\mathcal{A})$ of rank $\leq k+1$, and $\Sigma_C \omega = 0$.

This definition leads to a refinement of Theorem 3.5.

Proposition 3.8. *Let $\mathcal{A}$ be a central arrangement, of rank $r \geq 3$. If $\omega \in A^1_{\mathbb{C}}(\mathcal{A})$ is k–nonresonant, $1 \leq k < r-1$, then*

$$\dim_{\mathbb{C}} H_q(M_{\mathcal{A}}, \mathbb{C}_{\rho_\omega}) = \beta_q(\mathcal{A}, \omega)\,, \forall\, q \leq k\,. \tag{3.1}$$

Proof. Pick a subspace U, $(k+2)$–dimensional and $\mathcal{L}_{k+1}(\mathcal{A})$–generic. By Proposition 2.6, we may replace $\mathcal{A}$ by $\mathcal{A}^U$ in (3.1) above. Note also that $\mathrm{rk}(\mathcal{A}^U) = k+2$. Once we have checked that $\omega \in A^1_{\mathbb{C}}(\mathcal{A}^U)$ is nonresonant, our claim follows from Theorem 3.5.

To do this, we start by observing that the correspondence $X \rightsquigarrow X \cap U$ gives a bijection between $\mathcal{L}(\mathcal{A})$ and $\mathcal{L}(\mathcal{A}^U)$, in rank $\leq k+1$. This is a direct consequence of the fact that U is $\mathcal{L}_{k+1}(\mathcal{A})$–generic. Moreover, it is straightforward to verify that this bijection is order and rank preserving, and induces a bijection $\mathcal{A}_X \xrightarrow{\sim} \mathcal{A}^U_{X\cap U}$, if $\mathrm{rk}(X) \leq k+1$.

To check that the bijection also preserves dense elements, it is enough to recall from [6, Theorem 2] that $X \in \mathcal{L}(\mathcal{A})$ is dense if and only if $(1+t)^2$ does not divide the Poincaré polynomial $P_{\mathcal{A}_X}(t)$.

Finally, just note that the partial nonresonance conditions for $\mathcal{A}$ coincide with the nonresonance conditions for $\mathcal{A}^U$, in rank $\leq k+1$, while the remaining nonresonance condition(s), for the center of $\mathcal{A}^U$, take(s) a stronger form in $\mathcal{A}$; compare Definitions 3.4 and 3.7. □

3.9. We would like to apply the above proposition to $\frac{\mathbf{1}}{d} := \sum_{H\in\mathcal{A}} \frac{a_H}{d}$. But the 1–nonresonance condition is clearly violated, as soon as X has rank 2, $m_X > 2$ and $d \mid m_X$; see Example 3.3(ii). This prompts the next definition.

Definition 3.10. An element $\omega \in A^1_{\mathbb{C}}(\mathcal{A})$ is *k–admissible* if there is $\alpha \in \mathbb{Z}^n$ such that $\omega + \alpha$ is k–nonresonant.

Corollary 3.11. *Assume* $\mathrm{rk}(\mathcal{A}) \geq 3$. *Let $\rho \in \mathbb{T}_{\mathcal{A}}$ be a rational character. If $\frac{\mathbf{k}}{d}$ is k–admissible, then $b_q(\mathcal{A}, \frac{\mathbf{k}}{d}) = \beta_q(\mathcal{A}, \frac{\mathbf{k}}{d} + \alpha)$, $\forall\, q \leq k$, where α is as in Definition 3.10.*

Unfortunately, there are simple nonadmissible examples.

Example 3.12. Let $\mathcal{A}$ be the full Coxeter arrangement A_{v-1}, corresponding to the complete unsigned graph on v vertices. When $v \geq 5$, $\frac{\mathbf{1}}{3}$ is not 1–admissible.

Assuming the contrary, we may find $\alpha_{ij} \in \mathbb{Z}$, $1 \leq i \neq j \leq v$, with the property that $\alpha_{ij} + \alpha_{jk} + \alpha_{ki} \geq 1$, for all distinct i, j, k, and $\Sigma_C \alpha = \frac{v(v-1)}{6}$. Summing the inequalities, we get $(v-2)\Sigma_C \alpha \geq \binom{v}{3}$. Therefore, all inequalities must be equalities. Solving the system for $v = 4$, we find out that necessarily $\alpha_{ij} = \alpha_{kl}$, if i, j, k, l are distinct. If $v \geq 5$, this implies that $\alpha_{ij} = \alpha_{jk} = \alpha_{ki} = \frac{1}{3}$, for all distinct i, j, k, a contradiction.

Nevertheless, Corollary 3.11 turns out to be very useful to obtain vanishing results. To make this statement precise, we need the following definitions. For a given arrangement $\mathcal{A}$ and for each $k \geq 2$, set

$$\mathbf{m}_k(\mathcal{A}) := \{m_X \mid X \in \mathcal{L}(\mathcal{A}) \quad \text{dense} \quad \text{and} \quad 2 \leq \operatorname{rk}(X) \leq k\}\,. \tag{3.2}$$

For a fixed hyperplane $K \in \mathcal{A}$, set also

$$\mathbf{m}_k^K(\mathcal{A}) := \{m_X \mid X \in \mathcal{L}(\mathcal{A}) \quad \text{dense}\,, \quad X \not\subset K \quad \text{and} \quad 2 \leq \operatorname{rk}(X) \leq k\}\,. \tag{3.3}$$

We may now state our result.

Theorem 3.13. *Let $\mathcal{A}$ be a central arrangement of rank $r \geq 3$, with n hyperplanes, and $1 \leq k < r-1$. If $1 \neq d \mid n$ is such that d does not divide m, for any $m \in \mathbf{m}_{k+1}^K(\mathcal{A})$, for some $K \in \mathcal{A}$, then $b_{qd}(\mathcal{A}) = 0$, for all $q \leq k$.*

Proof. Define $\alpha \in \mathbb{Z}^n$ by: $\alpha_H = 0$ (for $H \neq K$), and $\alpha_K = \frac{n}{d}$. We claim that $\omega := \frac{\mathbf{1}}{d} - \alpha$ is k–nonresonant. Plainly, $\Sigma_C \omega = 0$. The rank one nonresonance conditions involve $\Sigma_H \omega$, which equals either $\frac{1}{d}$ (if $H \neq K$), which is not an integer, or $\frac{1-n}{d} < 0$ (if $H = K$). For X dense, $X \not\subset K$, with $2 \leq \operatorname{rk}(X) \leq k+1$, $\Sigma_X \omega = \frac{m_X}{d} - \Sigma_X \alpha$ cannot be an integer, since d does not divide m_X. If $X \subset K$, then $\Sigma_X \omega = (m_X - n)/d \leq 0$. Thus, the k–nonresonance claim is established.

Hence, Proposition 3.8 applies, and guarantees that $b_q(\mathcal{A}, \mathbf{1}/d) = \beta_q(\mathcal{A}, \omega)$, for all $q \leq k$. Our next claim is that $\beta_q(\mathcal{A}, \omega) = 0$, if $q \leq k$. This may be seen by using [26, Theorem 4.1(ii)], as follows. Pick a $(k+2)$–subspace U, which is $\mathcal{L}_{k+1}(\mathcal{A})$–generic. Due to Proposition 2.6 (2), we may replace $\mathcal{A}$ by $\mathcal{A}^U$.

Let us check now, for $\mathcal{A}^U$, the hypotheses needed in the abovementioned result of Yuzvinsky. As we have seen before, $\Sigma_C \omega = 0$. The remaining conditions involve $\Sigma_X \omega$, for $X \in \mathcal{L}(\mathcal{A}^U)$ with $1 \leq \operatorname{rk}(X) \leq k+1$. Recall from the proof of Proposition 3.8 that these elements X are identified with the elements X from $\mathcal{L}(\mathcal{A})$ of rank at most $k+1$; moreover, $\Sigma_X \omega$ takes the same value in $\mathcal{A}^U$ as in $\mathcal{A}$.

There are two cases to be considered. If $X \subset K$, then $\Sigma_X \omega = (m_X - n)/d < 0$ (since $\mathcal{A}^U_{X\cap U} \neq \mathcal{A}^U$). Otherwise, $\Sigma_X \omega = m_X/d > 0$. In both cases, $\Sigma_X \omega \neq 0$, and we are done.

We may conclude by deducing inductively from $b_q(\mathcal{A}, \mathbf{1}/d) = 0$, for $q \leq k$, that $b_{qd}(\mathcal{A}) = 0$, for $q \leq k$, as stated, via (2.2). $\square$

3.14. Our theorem above complements a similar result obtained by Libgober, who proved in [15], with a different method, that the non-divisibility conditions for all $X \in \mathcal{L}(\mathcal{A})$, dense, with rank between 2 and $k+1$, and contained in some $K \in \mathcal{A}$, imply the same conclusion. Either vanishing criterion may be used to deduce the following consequence, that led us to the formula from Theorem A (4).

Corollary 3.15. *Let $\mathcal{A}$ be an arbitrary subarrangement, with n hyperplanes and of rank ≥ 3, of a Coxeter arrangement. If $d \mid n$ and $d \notin \{1, 2, 3, 4, 5\}$, then $b_{1d}(\mathcal{A}) = 0$.*

Proof. We know that $\mathcal{A} \subset T$, where T is a full Coxeter arrangement and $\mathrm{rk}(T) \geq 3$. Pick any rank two element $X \in \mathcal{L}(\mathcal{A})$. Plainly, $\mathcal{A}_X \subset T_X$. Inspecting the tables from [18], we conclude that $m_X \leq 5$. Therefore, the $\mathbf{m}_2$–list of $\mathcal{A}$ defined in (3.2) is contained in $\{3, 4, 5\}$. Our assertion becomes then a direct consequence of Theorem 3.13. $\square$

4. Mod p Aomoto complexes of graphic arrangements ($p \neq 3$)

4.1. We will use the following terminology and notation. Denote by $[\ell]$ the set of points $\{1, \ldots, l\}$. We say that Γ is a graph in $[\ell]$ if the set of edges of Γ decomposes, $\mathsf{E}(\Gamma) = \mathsf{E}_1(\Gamma) \sqcup \mathsf{E}_2(\Gamma)$, where $\mathsf{E}_1(\Gamma) \subset [\ell]$ is the set of loops and $\mathsf{E}_2(\Gamma)$, the set of signed edges, consists of elements of the form ij^ϵ, with $\{i \neq j\} \subset [\ell]$ and $\epsilon \in \{\pm 1\}$.

Definition 4.2. If Γ is a graph in $[\ell]$, we denote by $\overline{\Gamma}$ the ordinary simplicial graph with set of edges $\mathsf{E}(\overline{\Gamma}) = \{ij := \{i \neq j\} \mid \exists\, \epsilon$ such that $ij^\epsilon \in \mathsf{E}_2(\Gamma)\}$. We also denote by $\mathsf{V}(\Gamma) = \mathsf{V}(\overline{\Gamma}) := \{i \in [\ell] \mid \exists\, e \in \mathsf{E}(\overline{\Gamma})$ such that $i \in e\}$, the set of vertices of Γ ($\overline{\Gamma}$).

Here is the definition of the arrangement associated to a graph.

Definition 4.3. Let Γ be a graph in $[\ell]$. We denote by $\mathcal{A}(\Gamma)$ the arrangement in $\mathbb{C}^l$, with hyperplanes given by the equations $x_i + \epsilon x_j = 0$, for each signed edge $ij^\epsilon \in \mathsf{E}_2(\Gamma)$, and $x_i = 0$, for each loop $i \in \mathsf{E}_1(\Gamma)$.

Example 4.4. Complete graphs.

(i) If Γ is the complete unsigned graph on l vertices, then $\mathcal{A}(\Gamma)$ is the braid arrangement of rank $l-1$, with defining equation $\prod_{1\le i<j\le l}(x_i - x_j) = 0$.
(ii) If Γ is the complete signed graph on l vertices, then $\mathcal{A}(\Gamma)$ is the arrangement of hyperplanes corresponding to the Coxeter group D_l, with defining equation $\prod_{1\le i<j\le l}(x_i \pm x_j) = 0$.
(iii) If in addition to that the graph has a loop at each vertex, then we get the arrangement corresponding to the Coxeter group B_l, defined by $\prod_{i=1}^{l} x_i \cdot \prod_{1\le i<j\le l}(x_i \pm x_j) = 0$.

4.5. **Rank 2 elements in a graphic arrangement.** In what follows we will refer mainly to graphic arrangements, so it will be convenient to use the label Γ for objects associated to the arrangement $\mathcal{A}(\Gamma)$; for instance, the lattice $\mathcal{L}(\mathcal{A}(\Gamma))$ is denoted simply by $\mathcal{L}(\Gamma)$, and so on.

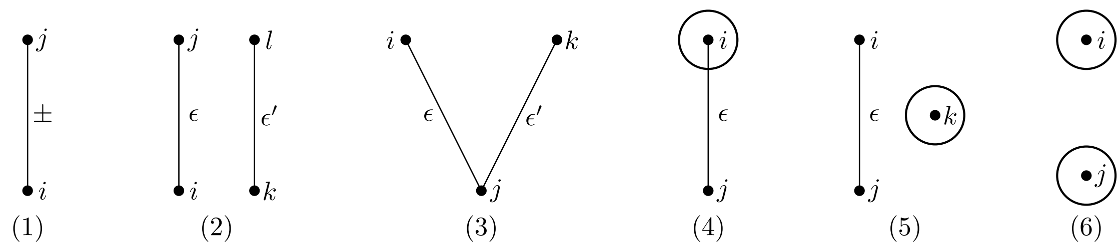


FIGURE 2. Pairs of edges

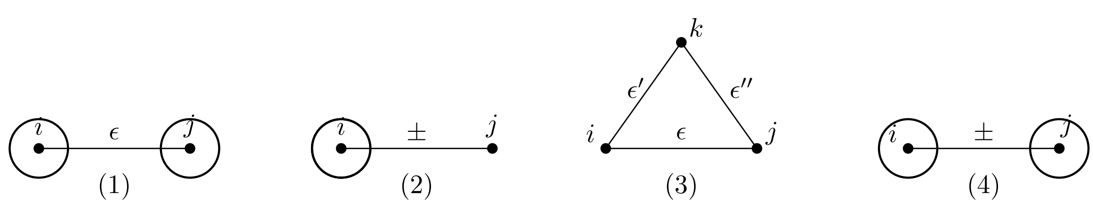


FIGURE 3. Dense elements

For reasons that will become clear from subsection §4.10 on, we draw up a complete inventory of rank 2 elements $X \in \mathcal{L}(\Gamma)$, by representing the subgraphs corresponding to the associated subarrangements, $\mathcal{A}_X(\Gamma)$. See figures 2 and 3.

Remark 4.6. Recall from §3.1 that m_X denotes the number of hyperplanes in the subarrangement $\mathcal{A}_X$, for $X \in \mathcal{L}(\mathcal{A})$. In Figure 2, $m_X = 2$, while $m_X = 3$ or 4, in Figure 3. In Figure 2, the configuration (3) means that $ik^{-\epsilon\epsilon'} \notin \mathsf{E}_2(\Gamma)$. In Figure 2(4), ij is a *simple* edge of $\overline{\Gamma}$ (identified with the corresponding edge, ij^{ϵ}, of Γ), that is, $ij^{-\epsilon} \notin \mathsf{E}_2(\Gamma)$. In Figure 3(2), ij is a *double* edge of $\overline{\Gamma}$ (identified with the corresponding pair of edges in Γ, $ij^{\pm}$). In Figure 3(3), the signs on the edges must be such that $\epsilon\epsilon'\epsilon'' = -1$. Such a triangle is called *negative* (otherwise the triangle is called *positive*).

4.7. **Weighted graphs.** An element $\eta \in A^1_{\Bbbk}(\Gamma)$, $\Bbbk$ a field, may be viewed as a collection of *weights*, that is, a set of coefficients, $\eta_k \in \Bbbk$, one for each $k \in \mathsf{E}_1(\Gamma)$, and $\eta^{\epsilon}_{ij} \in \Bbbk$, one for each $ij^{\epsilon} \in \mathsf{E}_2(\Gamma)$. If $\Bbbk = \mathbb{F}_p$, we will abbreviate $\mathbb{F}_p$ by p, when referring to the coefficient field; for instance, $A^1_p(\Gamma) := A^1_{\mathbb{F}_p}(\Gamma)$.

Remark 4.8. Denote by $Z_p(\Gamma)$ the set of 1–cocycles in $(A^*_p(\Gamma), \mu_1)$ (see (1.2)). Then $\beta_p(\Gamma) = 0$ if and only if the weights of η are constant on $\mathsf{E}(\Gamma)$, for any $\eta \in Z_p(\Gamma)$.

The following well-known result will be extensively used in computing $\beta_p(\Gamma)$, for p a prime.

Lemma 4.9. *Let $\mathcal{A}$ be an arbitrary central arrangement, p be a prime. If $\eta \in A^1_p(\mathcal{A})$, $\eta = \sum_{H\in\mathcal{A}} \eta_H a_H$, then $\eta\omega_1 = 0$ if and only if one has*

$$\Sigma_X \eta := \sum_{H \supset X} \eta_H = 0, \quad if \quad p \mid m_X, \tag{4.1}$$

or

$$\eta_H = \eta_K, \ \ \forall\ H \neq K \ \ \in \mathcal{A}_X, \ \ if\ p \ \ \nmid m_X\,, \tag{4.2}$$

for every rank 2 element $X \in \mathcal{L}_2(\mathcal{A})$.

Proof. See for instance [16, Lemma 3.3]. □

4.10. **Graphic arrangements at primes different from 3.** We will need to compute the numbers $\beta_p(\Gamma)$, for arbitrary Γ and p, when rk $\mathcal{A}(\Gamma) > 2$. We end this section by showing that these numbers are zero, for $p \neq 3$.

Lemma 4.11. *If $p \neq 2,\ 3$, then $\beta_p(\Gamma) = 0$.*

Proof. Let $H \neq K$ be arbitrary hyperplanes in $\mathcal{A}(\Gamma)$. Set $X = X(H, K) := H \cap K \in \mathcal{L}_2(\Gamma)$. Consider $\eta \in Z_p(\Gamma)$, $\eta = \sum_{H\in\mathcal{A}(\Gamma)} \eta_H a_H$. By inspecting Figures 2 and 3 from subsection 4.5, we see that the condition $p \nmid m_X$ from (4.2) is satisfied, so $\eta_H = \eta_K$, as needed (see Remark 4.8). □

The same argument actually proves the following analog of Theorem 3.13.

Proposition 4.12. *Let $\mathcal{A}$ be an arbitrary central arrangement. If a prime p does not divide m, for any $m \in \mathbf{m}_2(\mathcal{A})$, then $\beta_{1p}(\mathcal{A}) = 0$.*

Corollary 4.13. *Let $\mathcal{A}$ be an arbitrary subarrangement, of rank ≥ 3, of an arbitrary Coxeter arrangement. Then $\beta_{1p}(\mathcal{A}) = 0$, for $p \notin \{2, 3, 5\}$.*

Proof. Recall from the proof of Corollary 3.15 that $\mathbf{m}_2(\mathcal{A}) \subset \{3, 4, 5\}$. $\square$

Proposition 4.14. *Assume* rk $\mathcal{A}(\Gamma) > 2$. *Then $\beta_2(\Gamma) = 0$.*

Proof. Consider an arbitrary element $\eta \in Z_2(\Gamma)$. We have to show that $\eta_H = \eta_K$, $\forall H \neq K \in \mathcal{A}(\Gamma)$. Set $X = H \cap K \in \mathcal{L}_2(\Gamma)$. If $m_X \in \{2, 3\}$, then we are done, by resorting to Lemma 4.9.

Otherwise, $m_X = 4$, that is, the subarrangement $\mathcal{A}_X(\Gamma)$ is given by a subgraph of the type depicted in Figure 3(4), where say $i = 1$ and $j = 2$.

Then the weights of η on $\mathcal{A}_X(\Gamma)$ must satisfy

$$\eta^+_{12} + \eta^-_{12} + \eta_1 + \eta_2 = 0\,, \tag{4.3}$$

by Lemma 4.9. Since rk $\mathcal{A}(\Gamma) > 2$, there must be an edge e (of weight say a) in $\mathsf{E}(\Gamma)$, corresponding to a hyperplane that does not contain X.

Two cases may occur:

Case (a) There is an edge $e \in \mathsf{E}_2(\Gamma)$, different from $12^{\pm}$.

Subcase (a.1) Both endpoints of e are different from 1 and 2. In this case, figures 2(2) and 2(5) imply, via Lemma 4.9, that η has constant weight, equal to a, on $\mathcal{A}_X(\Gamma)$. In particular, $\eta_H = \eta_K$, as asserted.

Subcase (a.2) Otherwise, we may assume $e = 13^{\epsilon} \in \mathsf{E}_2(\Gamma)$. Then $\eta_2 = a$ (see figure 2(5) and Lemma 4.9). Moreover, $\eta^-_{12} = \eta^+_{12} = a$, as follows from figure 2(3) or figure 3(3), again by Lemma 4.9. We infer then from (4.3) that η has constant weight on $\mathcal{A}_X(\Gamma)$, and we are done.

Case (b) There are no other edges in $\mathsf{E}_2(\Gamma)$, except $12^{\pm}$, but there is a loop e in $\mathsf{E}_1(\Gamma)$, at $k \neq 1, 2$. Then $\eta^{\pm}_{12} = a$, and $\eta_1 = \eta_2 = a$, by Lemma 4.9 (see figure 2(5) and figure 2(6) respectively). $\square$

5. Mod 3 graphic Aomoto complexes

We analyze now what happens at the prime 3.

Proposition 5.1. *Assume* rk $\mathcal{A}(\Gamma) > 2$.

(1) *If $\beta_3(\Gamma) \neq 0$, then Γ must be one of the graphs from Figures 4 and 5.*
(2) *If Γ is exceptional, then $\beta_3(\Gamma) = 1$.*

5.2. **Preliminary lemmas.** The proof of Proposition 5.1 will occupy the rest of this section, where the coefficient field is understood to be $\mathbb{F}_3$.

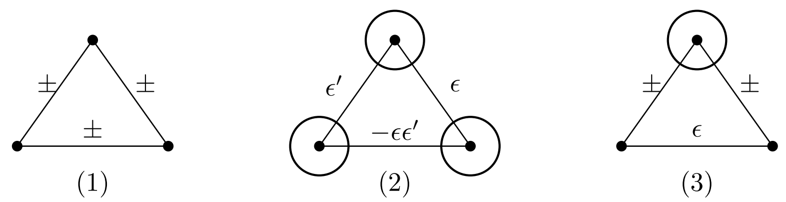


FIGURE 4. Exceptional graphs

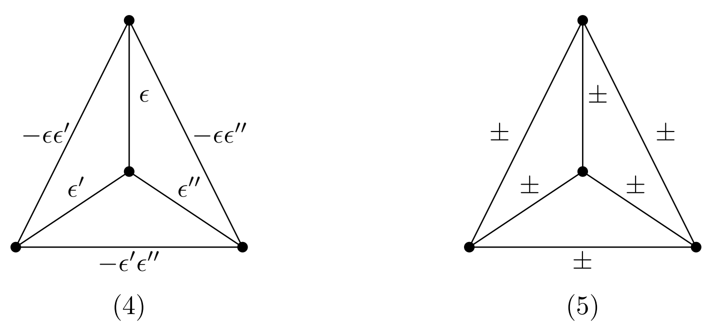
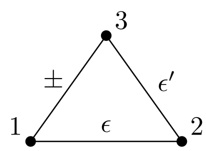


FIGURE 5. More exceptional graphs

Lemma 5.3. *Let $\Gamma' \subset \Gamma$ be a subgraph such that $\overline{\Gamma'}$ is a triangle. Assume that $\mathsf{E}_2(\Gamma')$ contains a simple edge of $\overline{\Gamma}$, and a double edge of $\overline{\Gamma}$. Assume also that Γ has no loops at the vertices of the triangle. If $\eta \in Z(\Gamma)$, then the weights of η are constant, on all edges of Γ'.*

Proof. Let the subgraph be as in the picture below. Here the edge 13 is double ($13^{\pm} \in \mathsf{E}_2(\Gamma')$), the edge 12 is simple ($12^{\epsilon} \in \mathsf{E}_2(\Gamma')$, $12^{-\epsilon} \notin \mathsf{E}_2(\Gamma)$), and $23^{\epsilon'}$ is one of the (at most two) edges from $\mathsf{E}_2(\Gamma')$ corresponding to $23 \in \mathsf{E}(\overline{\Gamma'})$. Denote η_{13}^{+} by a. We have to show that $\eta_{13}^{-} = \eta_{12}^{\epsilon} = \eta_{23}^{\epsilon'} = a$.



Since there are no Γ–loops in [3], we infer from figure 2(1) and Lemma 4.9 that $\eta_{13}^{+} = \eta_{13}^{-} = a$.

Set $\epsilon'' = \epsilon\epsilon'$. Then $\eta_{23}^{\epsilon'} = \eta_{13}^{\epsilon''} = a$, since $12^{-\epsilon} \notin \mathsf{E}_2(\Gamma)$ (see figure 2(3) and (4.2)). Next, we obtain from figure 3(3) and (4.1) that $\eta_{12}^{\epsilon} + \eta_{13}^{-\epsilon''} + \eta_{23}^{\epsilon'} = 0$. Therefore, $\eta_{12}^{\epsilon} + 2a = 0$, whence $\eta_{12}^{\epsilon} = a$. Consequently, all weights of η from the triangle above are equal to a. $\square$

The following definition will be convenient for our purposes: the *full subgraph* Γ' of Γ, determined by $\mathsf{V}' \subset \mathsf{V}(\Gamma)$, has edges $\mathsf{E}(\Gamma') = \mathsf{E}_2(\Gamma') := \{ij^{\epsilon} \in \mathsf{E}_2(\Gamma) \mid i, j \in \mathsf{V}'\}$.

Lemma 5.4. *Let Γ be a graph whose associated unsigned graph, $\overline{\Gamma}$, is complete on 4 vertices. If $\eta \in Z(\Gamma)$ has constant weight on $\mathsf{E}_2(\Gamma')$, where Γ' is a full subgraph of Γ on 3 vertices, then η has constant weight on $\mathsf{E}_2(\Gamma)$.*

Proof. Set $\mathsf{V}(\Gamma') = [3] \subset [4] = \mathsf{V}(\Gamma)$. We know that η has weight a, on $\mathsf{E}_2(\Gamma')$. Pick any edge $e = ij^{\epsilon} \in \mathsf{E}_2(\Gamma) \setminus \mathsf{E}_2(\Gamma')$. Clearly, $|\{i, j\} \cap [3]| = 1$, since Γ' is the full subgraph of Γ determined by [3]. Hence, we may find another edge, $f = kl^{\epsilon'} \in \mathsf{E}_2(\Gamma')$, such that $\{i, j\} \cap \{k, l\} = \emptyset$. Figure 2(2) and (4.2) together imply that $\eta_{ij}^{\epsilon} = \eta_{kl}^{\epsilon'} = a$. $\square$

Lemma 5.5. *Let Γ be a graph whose associated unsigned graph, $\overline{\Gamma}$, is complete on 4 vertices. If $\mathsf{E}_1(\Gamma) \neq \emptyset$, then the weights of η on Γ are constant, for any $\eta \in Z(\Gamma)$.*

Proof. Let $i \in \mathsf{E}_1(\Gamma)$ be an arbitrary loop, with weight a. We have to show that η has constant weight a on $\mathsf{E}_2(\Gamma)$. We may assume that $\mathsf{V}(\Gamma) = [4]$, and $i = 4$. (Indeed, if $i \notin \mathsf{V}(\Gamma)$, then figure 2(5) and Lemma 4.9 give the desired conclusion.)

Then $\eta_{ij}^{\epsilon} = a$, for any edge ij^{ϵ} of the full subgraph of Γ determined by [3] (use figure 2(5) and (4.2)). Lemma 5.4 yields then the desired conclusion. □

5.6. We begin the proof of Proposition 5.1(1) by a few preliminary remarks.

Remark 5.7. The assumption $\beta_3(\Gamma) \neq 0$ guarantees the existence of $\eta \in Z_3(\Gamma)$ with the property that the weights of η are not constant on $\mathcal{A}_X(\Gamma)$, for some $X \in \mathcal{L}_2(\Gamma)$. By Lemma 4.9(4.2), this forces $m_X = 3$. In other words, the subarrangement $\mathcal{A}_X(\Gamma)$ is represented by one of the first three graphs from Figure 3. So, there are three cases to be examined.

Remark 5.8. For each of the above configurations, the fact that two out of the three weights of η on $\mathcal{A}_X(\Gamma)$ are equal is equivalent to the fact that η has constant weight on $\mathcal{A}_X(\Gamma)$ (use (4.1) and remember that we are working modulo 3).

Remark 5.9. Due to our assumption on $\operatorname{rk} \mathcal{A}(\Gamma)$, there must be an edge $e \in \mathsf{E}(\Gamma)$, different from those of $\mathcal{A}_X(\Gamma)$.

5.10. **Proof of Proposition 5.1**(1)**.** We proceed to the analysis of the 3 abovementioned cases. Whenever possible without creating any ambiguity, we will omit the non-relevant signs of edges from $\mathsf{E}_2(\Gamma)$, to avoid making the exposition too heavy.

Case **(a)**: Suppose $\mathcal{A}_X(\Gamma)$ corresponds to a subgraph in Γ of the type described in Figure 3(3), with vertices labeled $i = 1, j = 2, k = 3$. We know that $\eta_{12} + \eta_{23} + \eta_{13} = 0$, from Lemma 4.9(4.1).

$(a.0)$ We may assume in case **(a)** that there is no edge in Γ of the form $e = ij$, with $\{i, j\} \cap [3] = \emptyset$. Indeed, otherwise figure 2(2) and (4.2) would imply that all weights of η on $\mathcal{A}_X(\Gamma)$ are equal to the weight of e, in contradiction with Remark 5.7.

Our discussion splits now, according to the number of vertices of Γ: either $|\mathsf{V}(\Gamma)| > 3$, or $|\mathsf{V}(\Gamma)| = 3$.

Case $(a.1)$: $|\mathsf{V}(\Gamma)| > 3$. We first claim that $ij \in \mathsf{E}(\overline{\Gamma})$, for every vertex j of Γ, $j \notin [3]$, and for all $i \in [3]$.

Indeed, denoting j by 4, we may resort to $(a.0)$ to assume that say 14 is an edge of Γ, with weight a. Then $\eta_{23} = a$ (by Lemma 4.9, applied to figure 2(2)).

If there is no edge in Γ connecting the vertices 2 and 4, or 3 and 4, we may apply Lemma 4.9 to figure 2(3) to deduce that $a = \eta_{12}$ (respectively $a = \eta_{13}$). Hence, η must be constant on $\mathcal{A}_X(\Gamma)$ (see Remark 5.8), which contradicts Remark 5.7. The claim is thus verified.

Again, there are two possibilities: either $|\mathsf{V}(\Gamma)| > 4$, or $|\mathsf{V}(\Gamma)| = 4$.

Subcase $(a.1.1)$: There are another vertices, say 4 and 5, of Γ. Due to the previous claim, $ij \in \mathsf{E}(\overline{\Gamma})$, for all $i \in [3]$ and $j = 4, 5$. It follows that $\eta_{12} = \eta_{35}$, $\eta_{13} = \eta_{24}$, and $\eta_{24} = \eta_{35}$, see figure 2(2) and (4.2). Therefore, $\eta_{12} = \eta_{13}$, contradicting again Remark 5.7, via Remark 5.8.

Subcase $(a.1.2)$: $\mathsf{V}(\Gamma) = [4]$. We already know that $\overline{\Gamma}$ is a complete graph.

If there exists a loop in Γ, we obtain a contradiction by applying Lemma 5.5. So, there are no loops in Γ. Now, if Γ contains a full subgraph on 3 vertices, having both simple and double edges, we may invoke lemmas 5.3 and 5.4 to infer that η has constant weight on Γ, which leads to the same contradiction as before. If not, it follows that Γ must be one of the graphs from Figure 5.

Indeed, this is clear if all edges of $\overline{\Gamma}$ are double. Otherwise, they must be all simple. Now, if there is a positive triangle in Γ, then η must have constant weight on it (see figure 2(3)). Again, Lemma 5.4 leads to a contradiction.

This completes the discussion of Case $(a.1)$.

Case $(a.2)$: $\mathsf{V}(\Gamma) = [3]$. In this case, we may suppose $\mathsf{E}_1(\Gamma) \subset [3]$ (otherwise, the equations provided by figure 2(5) would force η to have constant weight on $\mathsf{E}_2(\Gamma)$, in particular on $\mathcal{A}_X(\Gamma)$). In what follows, the discussion naturally splits according to the number of loops in Γ.

Subcase $(a.2.0)$: There are no loops in Γ. By virtue of Lemma 5.3, all edges must be double (see Remark 5.9). Thus, $\Gamma = D_3$, the first graph from Figure 4.

Subcase $(a.2.1)$: $|\mathsf{E}_1(\Gamma)| = 1$. Let 1 be the unique loop, with weight a. Then $a = \eta_{23}$ (by Lemma 4.9 and figure 2(5)). At this point, two possibilities may occur.

Subcase $(a.2.1')$: One of the edges 12 or 13 is simple. In this situation, we may apply Lemma 4.9 to figure 2(4), deducing that either $\eta_{12} = a$ or $\eta_{13} = a$, which contradicts Remark 5.7 (see Remark 5.8).

Subcase $(a.2.1'')$: Otherwise, both edges 12 and 13 are double. When all edges are double, Γ is entirely determined; a routine application of Lemma 4.9 shows then that $\beta_3(\Gamma) = 0$. When the edge 23 is simple, we obtain the graph from Figure 4(3).

Subcase $(a.2.2)$: $|\mathsf{E}_1(\Gamma)| = 2$, i.e., $\mathsf{E}_1(\Gamma)$ is say $\{1, 2\}$.

Subcase $(a.2.2')$: The edge 12 is double. Then it follows from Lemma 4.9 (4.2) that all 4 edges of the configuration from Figure 3(4) (where $ij = 12$) have the same weight, say a.

If one of the other edges, say 23, is simple, Lemma 4.9(4.2) may be applied to figure 2(4), to infer that $\eta_{23} = a$. By Remark 5.8, this contradicts Remark 5.7.

Finally, if all edges are double, a straightforward computation shows that $\beta_3(\Gamma) = 0$, like in subcase $(a.2.1'')$.

Subcase $(a.2.2'')$: The edge 12 is simple. This implies that $\eta_1 + \eta_2 + \eta_{12} = 0$ (see figure 3(1)). If the edge 13 is also simple, we obtain $\eta_1 = \eta_{13}$ (see figure 2(4)). We also get, by using figure 2(5), that $\eta_2 = \eta_{13}$. Putting these facts together, we deduce that $\eta_{12} = \eta_{13}$, a contradiction. If the edge 13 is double, then $\eta_1 + \eta_{13}^{+} + \eta_{13}^{-} = 0$ (see figure 3(2)), and $\eta_{13}^{\pm} = \eta_2$ (see figure 2(5)). Hence, the weights η_1, η_2, η_{12} and $\eta_{13}^{\pm}$ are all equal. In particular, $\eta_{12} = \eta_{13}^{\epsilon'}$, a contradiction.

Subcase $(a.2.3)$: $\mathsf{E}_1(\Gamma) = [3]$.

Subcase $(a.2.3')$: There is a simple edge, say 12, and a double edge, say 13. In this case, we have: $\eta_1 = \eta_3 = \eta_{13}^{\pm}$ (see figure 3(4)), and $\eta_3 = \eta_{12}$ (see figure 2(5)). These facts yield $\eta_{12} = \eta_{13}^{\epsilon'}$, a contradiction, as before.

Subcase $(a.2.3'')$: Either all edges are simple, i.e., Γ is the graph from Figure 4(2), or all edges are double, and then it is easy to see that $\beta_3(\Gamma) = 0$.

The analysis of case **(a)** is thus complete.

In the remaining two cases, $\mathcal{A}_X(\Gamma) = \mathcal{A}(\Gamma')$, where Γ' is a subgraph with shape described in figure 3(1)–(2), with say $ij = 12$. We begin by two remarks, valid in both these cases.

$(bc.1)$ We may assume that there is no edge ij in Γ disjoint from 12. Indeed, otherwise figures 2(2) and 2(5) would imply, via Lemma 4.9, that all weights of η on $\mathcal{A}_X(\Gamma)$ are equal to the weight of ij, a contradiction.

$(bc.2)$ We may also assume that $\mathsf{E}_2(\Gamma) \neq \mathsf{E}_2(\Gamma')$. If not, Remark 5.9 guarantees the existence of a loop of Γ away from [2], say 3. Using this time figures 2(5) and 2(6), we arrive again at a contradiction, as before.

Case **(b)**: $\mathcal{A}_X(\Gamma)$ corresponds to a subgraph in Γ of the type from figure 3(1). We know from Lemma 4.9 that $\eta_1 + \eta_2 + \eta_{12} = 0$.

It follows from $(bc.1) - (bc.2)$ above that we may suppose $13 \in \mathsf{E}(\overline{\Gamma})$. If $23 \notin \mathsf{E}(\overline{\Gamma})$, we infer from lemma 4.9 that $\eta_{12} = \eta_{13}$ and $\eta_2 = \eta_{13}$ (see figure 2, (3) and (5) respectively), thus contradicting Remark 5.7, via Remark 5.8. It follows that $13^{\epsilon'}, 23^{\epsilon''} \in \mathsf{E}_2(\Gamma)$, for some signs, ϵ' and ϵ''.

Subcase $(b+)$: The triangle 123 is positive. Then $\eta_{13}^{\epsilon'} = \eta_{23}^{\epsilon''}$ (see figure 2(3)). Moreover, $\eta_1 = \eta_{23}^{\epsilon''}$ and $\eta_2 = \eta_{13}^{\epsilon'}$ (see figure 2(5)). Hence, $\eta_1 = \eta_2$, a contradiction again.

Subcase $(b-)$: The triangle 123 is negative. If the weights of η on this triangle are not constant, we are back in case **(a)**, and we are done. Otherwise, denoting by a their common value, we may use figure 2(5) to deduce that η must have constant weight a on $\mathcal{A}_X(\Gamma)$, which contradicts our initial assumption from Remark 5.7.

The analysis of Case **(b)** is thus completed.

Case **(c)**: $\mathcal{A}_X(\Gamma)$ corresponds to a subgraph in Γ of the type from figure 3(2). Lemma 4.9 implies that $\eta_1 + \eta_{12}^+ + \eta_{12}^- = 0$.

As before, we know that either 13 or 23 is an edge of Γ, of weight say a. If they do not both belong to $\mathsf{E}(\overline{\Gamma})$, then figure 2(3) forces $\eta_{12}^{\pm} = a$, a contradiction. Consequently, we may find a negative triangle in Γ, with edges $13^{\epsilon'}$, $23^{\epsilon''}$ and 12^{ϵ}. Moreover, $\eta_1 = \eta_{23}^{\epsilon''}$ (see figure 2(5)).

If η has constant weight a on this triangle, then $\eta_1 = \eta_{12}^{\epsilon} = a$. Therefore, η must also have constant weight a on $\mathcal{A}_X(\Gamma)$, by Remark 5.8, which is impossible. Otherwise, we are again back in case **(a)**, and we are done.

This finishes the proof of Proposition 5.1(1).

5.11. Proposition 5.1(2) will follow from the next two lemmas.

Lemma 5.12. $\beta_3(D_3) = \beta_3(D_4) = 1$.

Proof. Direct computation, using Lemma 4.9. □

Lemma 5.13. *The exceptional graphic arrangements from Figures 4(2)-(3) and 5(4) are lattice–isotopic to D_3.*

Proof. We begin with the simplest case: the graph Γ from figure 5(4). By a convenient change of signs of the variables from $\mathbb{C}^4$, we can transform $\mathcal{A}(\Gamma)$ into $A_3 = D_3$. Similarly, we may assume that $\epsilon = \epsilon' = -1$, for the graphic arrangement $\mathcal{A}(\Gamma)$ from figure 4(2); by an obvious linear change of coordinates, we can finally make $\mathcal{A}(\Gamma)$ projectively equivalent, hence lattice–isotopic, to D_3.

By a preliminary change of signs, the last arrangement $\mathcal{A}(\Gamma)$ (see figure 4(3)) becomes defined by the equation $x_1(x_1 \pm x_2)(x_1 \pm x_3)(x_2 - x_3) = 0$. Next, we make the change of variables $x_1 = z_2 + z_3$; $x_1 + x_2 = z_1 + z_3$; $x_1 + x_3 = z_1 + z_2$. We arrive at a defining equation that corresponds to the value $t = -1$ in the family below (where $t \neq 1$)

$$(z_1 + z_2)(z_1 + z_3)(z_2 \pm z_3)[(z_1 - z_2) + t(z_2 + z_3)][(z_1 - z_3) + t(z_2 + z_3)] = 0\,.$$

It is straightforward to see that this family defines a lattice–isotopy from $\mathcal{A}(\Gamma)$ to D_3. □

6. Proof of Theorems A, B and C

We need one more ingredient: modular inequalities.

6.1. These inequalities may be formulated for arbitrary connected CW–spaces of finite type, M, endowed with a 1–marking, that is, a distinguished $\mathbb{Z}$–basis of $H_1(M)$. The marking allows us to extend Definition 2.4 verbatim, to this more general context, as well as the definition of $b_q(M, \mathbf{k}/d)$.

Consider next the prime field $\Bbbk = \mathbb{F}_p$. In the presence of the marking, we may speak about the element $\omega_{\mathbf{k}} \in H^1(M, \mathbb{F}_p)$, defined by taking the mod p reduction of $\mathbf{k}$. Hence, there is an associated Aomoto complex, $(H^\bullet(M, \mathbb{F}_p), \mu_{\mathbf{k}})$, defined exactly as in (1.2), leading to the numbers $\beta_{qp}(M, \mathbf{k})$; see (1.3). When $M = M_{\mathcal{A}}$ is an arrangement complement, $\beta_{qp}(M_{\mathcal{A}}, \mathbf{1}) = \beta_{qp}(\mathcal{A})$.

Theorem 6.2 ([20])**.** *Assume that the connected, finite type, 1–marked CW–space M has torsion–free integral homology. Let ρ be a rational local system on M, with denominator $d = p^s$, where p is prime and $s \geq 1$. Then*

$$b_q(M, \mathbf{k}/d) \leq \beta_{qp}(M, \mathbf{k})\,, \forall\, q\,.$$

This extends a result from [3], where M is an arrangement complement, and $s = 1$.

Corollary 6.3. *Let $\mathcal{A}$ be a central arrangement of n hyperplanes, and p be a prime such that $d := p^s$ divides n. If $\beta_{qp}(\mathcal{A}) = 0$, for $q \leq k$, then $b_{qd}(\mathcal{A}) = 0$, for $q \leq k$.*

Proof. By Theorem 6.2, $b_q(\mathcal{A}, \mathbf{1}/d) = 0$, for $q \leq k$. Hence, $b_{qd}(\mathcal{A}) = 0$, for $q \leq k$, by (2.2) and induction. □

6.4. **Proof of Theorem A.**

Part (1). Use figure 3 to infer that the $\mathbf{m}_2$–list of $\mathcal{A}(\Gamma)$ from (3.2) must be contained in $\{3, 4\}$. Therefore, Theorem 3.13 implies that $b_d(\Gamma) = 0$, if $d \neq 2, 3, 4$. For $d = 2$ or 4, recall from Proposition 4.14 that $\beta_2(\Gamma) = 0$, and use Corollary 6.3 to obtain again the vanishing of $b_d(\Gamma)$, as asserted.

Part (2). Follows from Lemma 4.11 and Proposition 4.14.

Part (3). By inspecting the graphs from Figures 4 and 5, we deduce from Proposition 5.1, in conjunction with Corollary 6.3, that either Γ is not exceptional, and then $\beta_3(\Gamma) = 0$, hence $b_3(\Gamma) = \beta_3(\Gamma) = 0$ for $n \equiv 0 \pmod 3$, or Γ is exceptional, and then $n \equiv 0 \pmod 3$ and $\beta_3(\Gamma) = 1$. Therefore, the proof of Part (3) is reduced, via Lemma 5.13, to checking that $b_3(D_3) = b_3(D_4) = 1$.

This in turn may be easily done by using the Deligne method, as follows. Choose integers a, b and c such that $a+b+c = -1$. Next, set $\alpha^{\pm}_{12} = \alpha^{\pm}_{34} = a$, $\alpha^{\pm}_{13} = \alpha^{\pm}_{24} = b$, and $\alpha^{\pm}_{23} = \alpha^{\pm}_{14} = c$. View $\{\alpha^{\pm}_{ij}\}_{1 \leq i < j \leq v}$ as an element $\alpha \in A^1_{\mathbb{Z}}(D_v)$, for $v = 3, 4$. It is easy to verify that $\frac{\mathbf{1}}{3} + \alpha \in A^1_{\mathbb{C}}(D_v)$ is 1–nonresonant, in the sense of Definition 3.7.

Hence, Proposition 3.8 applies and gives that $b_3(D_v) = b_1(D_v, \frac{\mathbf{1}}{3}) = \beta_1(D_v, \frac{\mathbf{1}}{3} + \alpha)$. The Aomoto Betti number $\beta_1(D_v, \frac{\mathbf{1}}{3} + \alpha)$ is then computed directly from the definition (2.3), by easy linear algebra, as explained in [16, Lemma 3.3].

Part (4). Let us inspect the equivariant decomposition of $H_1(F_\Gamma, \mathbb{Q})$ from (1.1). As recalled in the Introduction, the divisor $d = 1$ contributes with exponent $n-1$. No other divisors can contribute, excepting $d = 3$, by Part (1); at the same time, $\beta_2(\Gamma) = \beta_5(\Gamma) = 0$, by Part (2). Finally, $b_3(\Gamma) = \beta_3(\Gamma)$, by Part (3).

6.5. **Proof of Theorem B.** We know from §6.4 above that

$$H_1(F_\Gamma, \mathbb{Q}) = \big(\frac{\mathbb{Q}[t]}{t-1}\big)^{n-1} \oplus \big(\frac{\mathbb{Q}[t]}{t^2+t+1}\big)^{\beta_3(\Gamma)}.$$

Theorem B follows then from Proposition 5.1 and Lemma 5.13.

6.6. **Proof of Theorem C.** Theorem 6.2 predicts inequalities

$$b_1(\mathcal{A}(\Gamma), \mathbf{1}/p^s) \leq \beta_p(\Gamma), \quad \text{for} \quad s \geq 1, \tag{6.1}$$

at each prime p. We have to show that they all are actually equalities, if $s = 1$.

In rank ≥ 3, this follows from Theorem A(2)–(3).

This is equally true for an arbitrary rank 2 arrangement $\mathcal{A}$. Indeed, in this case one knows that

$$b_1(\mathcal{A}, \mathbf{1}/d) = \begin{cases} 0, & \text{if } d \nmid n; \\ n-2, & \text{if } d \mid n, \end{cases} \tag{6.2}$$

where $n = |\mathcal{A}|$ and $d \neq 1$, see for instance [24, Example 10.1]. As an immediate consequence of Lemma 4.9, we also have

$$\beta_p(\mathcal{A}) = \begin{cases} 0, & \text{if } p \nmid n; \\ n-2, & \text{if } p \mid n, \end{cases} \tag{6.3}$$

for every prime p. Our assertion follows then by comparing (6.2) (for $d = p$) and (6.3).

The proof of Theorem C is thus completed.

Remark 6.7. When $n = |\mathcal{A}|$ is prime, equations (6.2) and (6.3) above also show that the inequality (6.1) may well be strict, if $s > 1$.

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